

ESTIMATES ON MODULATION SPACES FOR SCHRÖDINGER EVOLUTION OPERATORS WITH QUADRATIC AND SUB-QUADRATIC POTENTIALS

KEIICHI KATO, MASAHARU KOBAYASHI AND SHINGO ITO

ABSTRACT. In this paper we give new estimates for the solution to the Schrödinger equation with quadratic and sub-quadratic potentials in the framework of modulation spaces.

1. INTRODUCTION

In this paper, we shall give estimates for the solution to the time dependent Schrödinger equation

$$(1) \quad \begin{cases} i\partial_t u(t, x) = -\frac{1}{2}\Delta u(t, x) + V(t, x)u(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n \end{cases}$$

in the framework of modulation spaces. Here $i = \sqrt{-1}$, $u(t, x)$ is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $V(t, x)$ is a real valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $u_0(x)$ is a complex valued function of $x \in \mathbb{R}^n$, $\partial_t u = \partial u / \partial t$ and $\Delta u = \sum_{i=1}^n \partial^2 u / \partial x_i^2$.

We shall highlight the case $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ and for all multi-indices α with $|\alpha| \geq 2$ or $|\alpha| \geq 1$ there exists $C_\alpha > 0$ such that

$$(2) \quad |\partial_x^\alpha V(t, x)| \leq C_\alpha, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$

There are a large number of works devoted to study the equation (1). Particularly, in the context of modulation spaces $M^{p,q}$, these types of issues were initiated in the works of Bényi-Gröchenig-Okoudjou-Rogers [1], Wang-Hudzik [16] and Wang-Zhao-Guo [17].

Theorem A. (Bényi-Gröchenig-Okoudjou-Rogers [1]) Let $1 \leq p, q \leq \infty$ and $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Suppose $V(t, x) = 0$. Then there exists a positive constant C such that

$$\|u(t, \cdot)\|_{M_{\varphi_0}^{p,q}} \leq C(1 + |t|)^{n/2} \|u_0\|_{M_{\varphi_0}^{p,q}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

for all $t \in \mathbb{R}$, where $u(t, x)$ is the solution of (1) with $u(0, x) = u_0(x)$.

Date: December 23, 2012.

2010 Mathematics Subject Classification. Primary 35Q41; Secondary 42B35.

Key words and phrases. Schrödinger operator, Modulation space, Wave packet transform.

Theorem B. (Wang-Hudzik [16]) Let $2 \leq p \leq \infty$, $1 \leq q \leq \infty$, $1/p + 1/p' = 1$ and $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Suppose $V(t, x) = 0$. Then there exists positive constants C and C' such that

$$\|u(t, \cdot)\|_{M_{\varphi_0}^{p,q}} \leq C(1 + |t|)^{-n(1/2-1/p)} \|u_0\|_{M_{\varphi_0}^{p',q}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

and

$$\|u(t, \cdot)\|_{M_{\varphi_0}^{p,q}} \leq C'(1 + |t|)^{n(1/2-1/p)} \|u_0\|_{M_{\varphi_0}^{p,q}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

for all $t \in \mathbb{R}$, where $u(t, x)$ is the solution of (1) with $u(0, x) = u_0(x)$.

The studies of this theme have been developed by a number of authors using a large variety of methods (see, for example, Bényi-Okoudjou [2], Cordero-Nicola [3], [4], Kobayashi-Sugimoto [12], Miyachi-Nicola-Rivetti-Tabacco-Tomita [13], Tomita [14], Wang-Huang [15]).

In our previous papers, we have the following estimates.

Theorem C. (Kato-Kobayashi-Ito [9], [10], [11]) Let $1 \leq p, q \leq \infty$ and $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$.

(i) Suppose $V(t, x) = 0$. Then

$$\|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{p,q}} = \|u_0\|_{M_{\varphi_0}^{p,q}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

holds for all $t \in \mathbb{R}$.

(ii) Suppose $V(t, x) = \pm \frac{1}{2}|x|^2$. Then

$$\|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{p,p}} = \|u_0\|_{M_{\varphi_0}^{p,p}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

holds for all $t \in \mathbb{R}$.

In (i) and (ii), $u(t, x)$ and $\varphi(t, x)$ denote the solutions of (1) with $u(0, x) = u_0(x)$ and $\varphi(0, x) = \varphi_0(x)$.

We remark that Theorem C covers Theorem A and B (see [9]).

To state our results, we define the Schrödinger operator of a free particle $e^{\frac{1}{2}it\Delta}$ by

$$(e^{\frac{1}{2}it\Delta}f)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1}[e^{-\frac{1}{2}it|\xi|^2} \mathcal{F}f(\xi)](x), \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Here we use the notation $\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix \cdot \xi} dx$ for the Fourier transform of f and $\mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^n} f(\xi)e^{ix \cdot \xi} d\vec{\xi}$ with $d\vec{\xi} = (2\pi)^{-n} d\xi$ for the inverse Fourier transform of f . The following theorems are our main results.

Theorem 1.1. Let $1 \leq p \leq \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and $T > 0$. Set $\varphi(t, x) = e^{\frac{1}{2}it\Delta}\varphi_0(x)$. If $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfies (2) for all multi-indices α with $|\alpha| \geq 2$, then there exists $C_T > 0$ such that

$$\|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{p,p}} \leq C_T \|u_0\|_{M_{\varphi_0}^{p,p}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

for all $t \in [-T, T]$, where $u(t, x)$ denotes the solution of (1) in $C(\mathbb{R}; L^2(\mathbb{R}^n))$ with $u(0, x) = u_0(x)$.

In the above theorem, we cannot expect to replace the $M^{p,p}$ norm with the $M^{p,q}$ norm. In fact, when $V(t, x) = \frac{1}{2}|x|^2$ we have

$$\|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{p,q}} = \| \|W_{\varphi_0} u_0(x \cos t - \xi \sin t, x \sin t + \xi \cos t)\|_{L_x^p} \|_{L_\xi^q}$$

and thus

$$\left\| u\left(\frac{\pi}{2}, \cdot\right) \right\|_{M_{\varphi(\frac{\pi}{2}, \cdot)}^{p,q}} = \| \|W_{\varphi_0} u_0(\xi, x)\|_{L_x^p} \|_{L_\xi^q}$$

(refer to [10]), but $\| \|W_{\varphi_0} u_0(\xi, x)\|_{L_x^p} \|_{L_\xi^q} \leq C \| \|W_{\varphi_0} u_0(x, \xi)\|_{L_x^p} \|_{L_\xi^q}$ does not hold generally. However, if we strengthen the assumption of V then we can replace the $M^{p,p}$ norm with $M^{p,q}$ norm.

Theorem 1.2. *Let $1 \leq p, q \leq \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and $T > 0$. Set $\varphi(t, x) = e^{\frac{1}{2}it\Delta} \varphi_0(x)$. If $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfies (2) for all multi-indices α with $|\alpha| \geq 1$, then there exists $C_T > 0$ such that*

$$\|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{p,q}} \leq C_T \|u_0\|_{M_{\varphi_0}^{p,q}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

for all $t \in [-T, T]$, where $u(t, x)$ denotes the solution of (1) in $C(\mathbb{R}; L^2(\mathbb{R}^n))$ with $u(0, x) = u_0(x)$.

Remark 1.3. In Theorem 1.1 and Theorem 1.2, we assume $V(t, x) \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ but, in fact, it is enough to assume $V(t, x)$ is C^2 -function in t and $C^{2[\frac{n}{2}]+4}$ -function in x .

This paper is organized as follows. In Section 2, we give some notations and recall the definitions and basic properties of wave packet transform and modulation spaces. In Section 3, we give some properties concerning the orbit of the classical mechanics corresponding to the Schrödinger equation (1). In Section 4, we prove Theorem 1.1. Finally, in Section 5, we prove Theorem 1.2.

2. PRELIMINARIES

2.1. Notations. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and a $m \times n$ matrix $A = (a_{ij})$, we denote

$$\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}, \quad \|x\|_\infty = \max_{1 \leq j \leq n} |x_j| \quad \text{and} \quad \|A\|_\infty = \max_{1 \leq j \leq m, 1 \leq k \leq n} |a_{jk}|.$$

For a real valued function $V \in C^1(\mathbb{R} \times \mathbb{R}^n)$, we put

$$\nabla_x V(t, x_1, \dots, x_n) = (\partial_{x_1} V(t, x_1, \dots, x_n), \dots, \partial_{x_n} V(t, x_1, \dots, x_n)).$$

Throughout this paper the letter C denotes a constant, which may be different in each occasion.

2.2. Wave Packet Transform. We recall the definition of the wave packet transform which is defined by Córdoba-Fefferman [5]. Wave packet transform is called short time Fourier transform or windowed Fourier transform in several literatures. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Then the wave packet transform $W_\varphi f(x, \xi)$ of f with the wave packet generated by a function φ is defined by

$$W_\varphi f(x, \xi) = \int_{\mathbb{R}^n} \overline{\varphi(y-x)} f(y) e^{-iy \cdot \xi} dy.$$

We call such φ window function. Let F be a function on $\mathbb{R}^n \times \mathbb{R}^n$. Then the (informal) adjoint operator W_φ^* of W_φ is defined by

$$W_\varphi^* F(x) = \iint_{\mathbb{R}^{2n}} F(y, \xi) \varphi(x-y) e^{ix \cdot \xi} dy d\xi$$

with $d\xi = (2\pi)^{-n} d\xi$. It is known that for $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ satisfying $\langle \psi, \varphi \rangle \neq 0$, we have the inversion formula

$$\frac{1}{\langle \psi, \varphi \rangle} W_\psi^* W_\varphi f = f, \quad f \in \mathcal{S}'(\mathbb{R}^n)$$

([8, Corollary 11.2.7]).

For the sake of convenience, we use the following notation

$$W_{\varphi(t, \cdot)} u(t, x, \xi) = W_{\varphi(t, \cdot)} [u(t, \cdot)](x, \xi) = \int_{\mathbb{R}^n} \overline{\varphi(t, y-x)} u(t, y) e^{-iy \cdot \xi} dy,$$

where $\varphi(t, x)$ and $u(t, x)$ are functions on $\mathbb{R} \times \mathbb{R}^n$.

2.3. Modulation Spaces. We recall the definition of modulation spaces $M^{p,q}$. Let $1 \leq p, q \leq \infty$ and $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Then the modulation space $M_\varphi^{p,q}(\mathbb{R}^n) = M^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the norm

$$\|f\|_{M_\varphi^{p,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |W_\varphi f(x, \xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} = \| \|W_\varphi f(x, \xi)\|_{L_x^p} \|_{L_\xi^q}$$

is finite (with usual modifications if $p = \infty$ or $q = \infty$).

The space $M_\varphi^{p,q}(\mathbb{R}^n)$ is a Banach space, whose definition is independent of the choice of the window function φ , i.e., $M_\varphi^{p,q}(\mathbb{R}^n) = M_\psi^{p,q}(\mathbb{R}^n)$ for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ ([6, Theorem 6.1]). This property is crucial in the sequel, since we choose a suitable window function φ to estimate the modulation space norm. If $1 \leq p, q < \infty$ then $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}$ ([6, Theorem 6.1]). We also note $L^2 = M^{2,2}$, and $M^{p_1, q_1} \hookrightarrow M^{p_2, q_2}$ if $p_1 \leq p_2, q_1 \leq q_2$ ([6, Proposition 6.5]). Let us define by $\mathcal{M}^{p,q}(\mathbb{R}^n)$ the completion of $\mathcal{S}(\mathbb{R}^n)$ under the norm $\|\cdot\|_{M^{p,q}}$. Then $\mathcal{M}^{p,q}(\mathbb{R}^n) = M^{p,q}(\mathbb{R}^n)$ for $1 \leq p, q < \infty$. Moreover, the complex interpolation theory for these spaces reads as follows: Let $0 < \theta < 1$ and $1 \leq p_i, q_i \leq \infty, i = 1, 2$. Set $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$, then $(\mathcal{M}^{p_1, q_1}, \mathcal{M}^{p_2, q_2})_{[\theta]} = \mathcal{M}^{p, q}$ ([6, Theorem 6.1], [15, Theorem 2.3]). We refer to [6] and [8] for more details.

3. KEY LEMMAS

The orbit of the classical mechanics corresponding to (1) is described by the system of ordinary differential equations

$$(3) \quad \begin{cases} \frac{d}{ds} f(s) = g(s), \\ \frac{d}{ds} g(s) = -(\nabla_x V)(s, f(s)), \end{cases}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}^n$ and $g : \mathbb{R} \rightarrow \mathbb{R}^n$ (see also Fujiwara [7]). So, we prepare two lemmas which give some properties of the solutions to the system of ordinary differential equations. Following lemma is used in the proof of Theorem 1.1.

Lemma 3.1. *Let $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfy (2) for all multi-indices α with $|\alpha| \geq 2$. Suppose that $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are solutions to (3) satisfying $f(t) = x$ and $g(t) = \xi$ and put*

$$M(s; t, x, \xi) = (w_{i,j})$$

$$= \begin{pmatrix} \frac{\partial f_1(s; t, x, \xi)}{\partial x_1} & \dots & \frac{\partial f_1(s; t, x, \xi)}{\partial x_n} & \frac{\partial f_1(s; t, x, \xi)}{\partial \xi_1} & \dots & \frac{\partial f_1(s; t, x, \xi)}{\partial \xi_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_n(s; t, x, \xi)}{\partial x_1} & \dots & \frac{\partial f_n(s; t, x, \xi)}{\partial x_n} & \frac{\partial f_n(s; t, x, \xi)}{\partial \xi_1} & \dots & \frac{\partial f_n(s; t, x, \xi)}{\partial \xi_n} \\ \frac{\partial g_1(s; t, x, \xi)}{\partial x_1} & \dots & \frac{\partial g_1(s; t, x, \xi)}{\partial x_n} & \frac{\partial g_1(s; t, x, \xi)}{\partial \xi_1} & \dots & \frac{\partial g_1(s; t, x, \xi)}{\partial \xi_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial g_n(s; t, x, \xi)}{\partial x_1} & \dots & \frac{\partial g_n(s; t, x, \xi)}{\partial x_n} & \frac{\partial g_n(s; t, x, \xi)}{\partial \xi_1} & \dots & \frac{\partial g_n(s; t, x, \xi)}{\partial \xi_n} \end{pmatrix}.$$

Then $\det M(s; t, x, \xi) = 1$ for all s, t, x and ξ .

It is easy to prove this lemma by the standard method, but we give the proof for reader's convenience in Appendix. Next lemma plays an important role in the proof of Theorem 1.2.

Lemma 3.2. *Let $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ be solutions to (3) with $f(t) = x$ and $g(t) = \xi$. If $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfies (2) for all multi-indices α with $|\alpha| \geq 1$, then there exist $C_1, C_2 > 0$ such that*

$$(4) \quad \frac{1}{\langle y - f(s; t, x, \xi) \rangle} \leq \frac{C_1(1 + |t - s|^2)}{\langle y - x + (t - s)\xi \rangle}$$

and

$$(5) \quad \frac{1}{\langle \eta - g(s; t, x, \xi) \rangle} \leq \frac{C_2(1 + |t - s|)}{\langle \eta - \xi \rangle}.$$

Proof. First, we show (4). Since $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ solve (3), we have

$$\begin{aligned}
 f(s; t, x, \xi) &= f(t; t, x, \xi) + \int_t^s g(\tau; t, x, \xi) d\tau \\
 &= x + \int_t^s \left(g(t; t, x, \xi) - \int_t^\tau (\nabla_x V)(\sigma, f(\sigma; t, x, \xi)) d\sigma \right) d\tau \\
 &= x + (s - t)\xi - \int_s^t \int_s^\sigma (\nabla_x V)(\sigma, f(\sigma; t, x, \xi)) d\tau d\sigma \\
 (6) \quad &= x + (s - t)\xi - \int_s^t (\sigma - s)(\nabla_x V)(\sigma, f(\sigma; t, x, \xi)) d\sigma.
 \end{aligned}$$

Since V satisfies (2) for all multi-indices α with $|\alpha| \geq 1$, we have

$$(7) \quad |(\partial_{x_j} V)(\sigma, f(\sigma; t, x, \xi))| \leq 2C$$

for $j = 1, 2, \dots, n$. By (6) and (7), we have

$$\begin{aligned}
 |y - x + (t - s)\xi| &\leq |y - f(s; t, x, \xi)| + |f(s; t, x, \xi) - x + (t - s)\xi| \\
 &\leq |y - f(s; t, x, \xi)| + \left| \int_s^t (\sigma - s)(\nabla_x V)(\sigma, f(\sigma; t, x, \xi)) d\sigma \right| \\
 &\leq |y - f(s; t, x, \xi)| + 2\sqrt{n}C \left| \int_s^t |\sigma - s| d\sigma \right| \\
 &= |y - f(s; t, x, \xi)| + \sqrt{n}C |t - s|^2.
 \end{aligned}$$

So, we have

$$\begin{aligned}
 \langle y - x + (t - s)\xi \rangle &\leq \{1 + 2(|y - f(s; t, x, \xi)|^2 + nC^2 |t - s|^4)\}^{1/2} \\
 &\leq \sqrt{2}(1 + \sqrt{n}C |t - s|^2) \langle y - f(s; t, x, \xi) \rangle.
 \end{aligned}$$

Putting $C_1 = \sqrt{2} \max\{1, \sqrt{n}C\}$, we obtain (4).

Next, we show (5). Since $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ solve (3), we have

$$\begin{aligned}
 g(s; t, x, \xi) &= g(t; t, x, \xi) - \int_t^s \nabla_x V(\sigma, f(\sigma; t, x, \xi)) d\sigma \\
 &= \xi + \int_s^t \nabla_x V(\sigma, f(\sigma; t, x, \xi)) d\sigma
 \end{aligned}$$

and thus,

$$\begin{aligned}
 |\eta - \xi| &\leq |\eta - g(s; t, x, \xi)| + |g(s; t, x, \xi) - \xi| \\
 &\leq |\eta - g(s; t, x, \xi)| + \left| \int_s^t (\nabla_x V)(\sigma, f(\sigma; t, x, \xi)) d\sigma \right| \\
 &\leq |\eta - g(s; t, x, \xi)| + 2\sqrt{n}C |t - s|.
 \end{aligned}$$

Hence, we have

$$\begin{aligned}\langle \eta - \xi \rangle &\leq \left\{ 1 + 2(|\eta - g(s; t, x, \xi)|^2 + 4nC^2|t - s|^2) \right\}^{\frac{1}{2}} \\ &\leq \sqrt{2}(1 + 2\sqrt{n}C|t - s|)\langle \eta - g(s; t, x, \xi) \rangle.\end{aligned}$$

Putting $C_2 = \sqrt{2} \max\{1, 2\sqrt{n}C\}$, we obtain (5). \square

4. PROOF OF THEOREM 1.1

We only consider the case $t \in [0, T]$. We can treat the case $t \in [-T, 0]$ in the same way. First, by using wave packet transform, we transform (1) into a first order partial differential equation and a lower order term. By integration by parts, we have

$$(8) \quad W_{\varphi(t, \cdot)}(i\partial_t u)(t, x, \xi) = i\partial_t W_{\varphi(t, \cdot)}u(t, x, \xi) + W_{i\partial_t \varphi(t, \cdot)}u(t, x, \xi)$$

and

$$\begin{aligned}(9) \quad &W_{\varphi(t, \cdot)}\left(\frac{1}{2}\Delta u\right)(t, x, \xi) \\ &= W_{\frac{1}{2}\Delta \varphi(t, \cdot)}u(t, x, \xi) + i\xi \cdot \nabla_x W_{\varphi(t, \cdot)}u(t, x, \xi) - \frac{|\xi|^2}{2}W_{\varphi(t, \cdot)}u(t, x, \xi),\end{aligned}$$

where $\varphi(t, x) = e^{\frac{1}{2}it\Delta}\varphi_0(x)$. Applying Taylor's theorem to $V(t, \cdot)$, we have, by integration by parts,

$$\begin{aligned}(10) \quad &W_{\varphi(t, \cdot)}(Vu)(t, x, \xi) \\ &= \int_{\mathbb{R}^n} \overline{\varphi(t, y - x)} \left(V(t, x) + \nabla_x V(t, x) \cdot (y - x) \right. \\ &\quad \left. + \sum_{j,k=1}^n (y_j - x_j)(y_k - x_k) V_{jk}(t, x, y) \right) u(t, y) e^{-iy \cdot \xi} dy \\ &= \{V(t, x) + i\nabla_x V(t, x) \cdot \nabla_\xi - \nabla_x V(t, x) \cdot x\} W_{\varphi(t, \cdot)}u(t, x, \xi) + Ru(t, x, \xi),\end{aligned}$$

where

$$\begin{aligned}(11) \quad &Ru(t, x, \xi) \\ &= \sum_{j,k=1}^n \int_{\mathbb{R}^n} \overline{\varphi(t, y - x)} V_{jk}(t, x, y) (y_j - x_j)(y_k - x_k) u(t, y) e^{-iy \cdot \xi} dy\end{aligned}$$

and

$$(12) \quad V_{jk}(t, x, y) = \int_0^1 \partial_{x_j} \partial_{x_k} V(t, x + \theta(y - x))(1 - \theta) d\theta.$$

Since $i\partial_t \varphi(t, x) + \frac{1}{2}\Delta \varphi(t, x) = 0$, we have

$$(13) \quad W_{i\partial_t \varphi(t, \cdot)}u(t, x, \xi) + W_{\frac{1}{2}\Delta \varphi(t, \cdot)}u(t, x, \xi) = 0.$$

Combining (8), (9), (10) and (13), the initial value problem (1) is transformed to

$$\begin{cases} \left(i\partial_t + i\xi \cdot \nabla_x - i\nabla_x V(t, x) \cdot \nabla_\xi - \frac{1}{2}|\xi|^2 - V(t, x) \right. \\ \quad \left. + \nabla_x V(t, x) \cdot x \right) W_{\varphi(t, \cdot)} u(t, x, \xi) - Ru(t, x, \xi) = 0, \\ W_{\varphi(0, \cdot)} u(0, x, \xi) = W_{\varphi_0} u_0(x, \xi). \end{cases}$$

By the method of characteristics, we obtain

$$(14) \quad W_{\varphi(t, \cdot)} u(t, x, \xi) = e^{-i \int_0^t h(s; t, x, \xi) ds} \left(W_{\varphi_0} u_0(f(0; t, x, \xi), g(0; t, x, \xi)) \right. \\ \left. - i \int_0^t e^{i \int_0^\tau h(s; t, x, \xi) ds} Ru(\tau, f(\tau; t, x, \xi), g(\tau; t, x, \xi)) d\tau \right),$$

where $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are solutions to (3) with $f(t) = x$ and $g(t) = \xi$, and

$$\begin{aligned} & h(s; t, x, \xi) \\ &= \frac{1}{2} |g(s; t, x, \xi)|^2 + V(s, f(s; t, x, \xi)) - \nabla_x V(s, f(s; t, x, \xi)) \cdot f(s; t, x, \xi). \end{aligned}$$

By taking L^p -norm with respect to x and ξ on both sides of (14), we have

$$(15) \quad \|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{p,p}} = \|W_{\varphi(t, \cdot)} u(t, x, \xi)\|_{L_{x, \xi}^p} \leq \|I_1\|_{L_{x, \xi}^p} + \int_0^t \|I_2\|_{L_{x, \xi}^p} d\tau,$$

where

$$(16) \quad I_1 = W_{\varphi_0} u_0(f(0; t, x, \xi), g(0; t, x, \xi)), \quad I_2 = Ru(\tau, f(\tau; t, x, \xi), g(\tau; t, x, \xi)).$$

Now we consider the change of variables $X = f(0; t, x, \xi)$ and $\Xi = g(0; t, x, \xi)$. From Lemma 3.1 and the implicit function theorem, we have

$$\left| \frac{\partial(x, \xi)}{\partial(X, \Xi)} \right| = 1.$$

So it follows that

$$(17) \quad \|I_1\|_{L_{x, \xi}^p} = \left(\iint_{\mathbb{R}^{2n}} |W_{\varphi_0} u_0(X, \Xi)|^p \left| \frac{\partial(x, \xi)}{\partial(X, \Xi)} \right| dX d\Xi \right)^{\frac{1}{p}} = \|u_0\|_{M_{\varphi_0}^{p,p}}.$$

On the other hand, from (11) and the inversion formula of wave packet transform for u , we have

$$\begin{aligned} Ru(t, x, \xi) &= \frac{1}{\|\varphi(t, \cdot)\|_{L^2}^2} \sum_{j,k=1}^n \iiint_{\mathbb{R}^{3n}} \varphi_{jk}(t, y-x) V_{jk}(t, x, y) \varphi(t, y-z) \\ &\quad \times W_{\varphi(t, \cdot)} u(t, z, \eta) e^{iy \cdot (\eta - \xi)} dz d\eta dy, \end{aligned}$$

where $\varphi_{jk}(t, y) = y_j y_k \overline{\varphi(t, y)}$. Take $N \in \mathbb{N}$ satisfying $2N > n$. From

$$(1 - \Delta_y)^N e^{iy \cdot (\eta - g(\tau; t, x, \xi))} = \langle \eta - g(\tau; t, x, \xi) \rangle^{2N} e^{iy \cdot (\eta - g(\tau; t, x, \xi))},$$

we have

$$\begin{aligned} & \|I_2\|_{L_{x, \xi}^p} \\ &= \|Ru(\tau, f(\tau; t, x, \xi), g(\tau; t, x, \xi))\|_{L_{x, \xi}^p} \\ &\leq \frac{1}{\|\varphi(\tau, \cdot)\|_{L^2}^2} \sum_{j,k=1}^n \left\| \iiint_{\mathbb{R}^{3n}} \left| (1 - \Delta_y)^N \left\{ \varphi_{jk}(\tau, y - f(\tau; t, x, \xi)) \right. \right. \right. \\ &\quad \times \left. \left. V_{jk}(\tau, f(\tau; t, x, \xi), y) \varphi(\tau, y - z) \right\} \right| \frac{|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)|}{\langle \eta - g(\tau; t, x, \xi) \rangle^{2N}} dz d\eta dy \Big\|_{L_{x, \xi}^p} \\ &\leq \frac{1}{\|\varphi(\tau, \cdot)\|_{L^2}^2} \sum_{j,k=1}^n \sum_{|\beta_1|+|\beta_2|+|\beta_3| \leq 2N} \left\| \iiint_{\mathbb{R}^{3n}} \left| \partial_y^{\beta_1} \varphi_{jk}(\tau, y - f(\tau; t, x, \xi)) \right. \right. \\ &\quad \times \left. \left. \partial_y^{\beta_2} V_{jk}(\tau, f(\tau; t, x, \xi), y) \partial_y^{\beta_3} \varphi(\tau, y - z) \right| \frac{|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)|}{\langle \eta - g(\tau; t, x, \xi) \rangle^{2N}} dz d\eta dy \right\|_{L_{x, \xi}^p}. \end{aligned}$$

Since $|\partial_y^{\beta_2} V_{jk}(\tau, f(\tau; t, x, \xi), y)| \leq C_{\beta_2}$ for $C_{\beta_2} > 0$, we have, by the change of variables $X = f(\tau; t, x, \xi)$ and $\Xi = g(\tau; t, x, \xi)$, Young's inequality and Lemma 3.1,

$$\begin{aligned} & \|I_2\|_{L_{x, \xi}^p} \\ &\leq \frac{1}{\|\varphi(\tau, \cdot)\|_{L^2}^2} \sum_{j,k=1}^n \sum_{|\beta_1|+|\beta_2|+|\beta_3| \leq 2N} C_{\beta_2} \left\{ \iint_{\mathbb{R}^{2n}} \left(\iint_{\mathbb{R}^{2n}} \frac{|\partial_y^{\beta_1} \varphi_{jk}(\tau, y - X)|}{\langle \eta - \Xi \rangle^{2N}} \right. \right. \\ &\quad \times \left. \left. \int_{\mathbb{R}^n} |\partial_y^{\beta_3} \varphi(\tau, y - z) W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)| dz d\eta dy \right)^p \left| \frac{\partial(x, \xi)}{\partial(X, \Xi)} \right| dX d\Xi \right\}^{\frac{1}{p}} \\ &\leq \frac{1}{\|\varphi(\tau, \cdot)\|_{L^2}^2} \sum_{j,k=1}^n \sum_{|\beta_1|+|\beta_2|+|\beta_3| \leq 2N} C_{\beta_2} \left\| \frac{\partial_y^{\beta_1} \varphi_{jk}(\tau, y)}{\langle \eta \rangle^{2N}} \right\|_{L_{y, \eta}^1} \|\partial_y^{\beta_3} \varphi(\tau, \cdot)\|_{L^1} \\ &\quad \times \|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)\|_{L_{z, \eta}^p} \\ (18) \\ &\leq C_T \|u(\tau, \cdot)\|_{M_{\varphi(\tau, \cdot)}^{p,p}} \end{aligned}$$

for $t \in [0, T]$ and $\tau \in [0, t]$. From (15), (17) and (18), we have

$$\|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{p,p}} \leq \|u_0\|_{M_{\varphi_0}^{p,p}} + C_T \int_0^t \|u(\tau, \cdot)\|_{M_{\varphi(\tau, \cdot)}^{p,p}} d\tau$$

for $t \in [0, T]$. Then Gronwall's inequality yields $\|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{p,p}} \leq C_T \|u_0\|_{M_{\varphi_0}^{p,p}}$ for $t \in [0, T]$. \square

5. PROOF OF THEOREM 1.2

We only consider the case $t \in [0, T]$, since we can treat the case $t \in [-T, 0]$ in the same way. First, we consider the case $(p, q) = (\infty, 1)$, next $(p, q) = (1, \infty)$ and finally general (p, q) .

In the proof of Theorem 1.1, we have already obtained

$$|W_{\varphi(t, \cdot)} u(t, x, \xi)| \leq |I_1| + \int_0^t |I_2| d\tau,$$

where I_1 and I_2 are defined by (16). Take $N \in \mathbb{N}$ satisfying $2N > n$. From

$$(1 - \Delta_y)^N e^{iy \cdot (\eta - g(0; t, x, \xi))} = \langle \eta - g(0; t, x, \xi) \rangle^{2N} e^{iy \cdot (\eta - g(0; t, x, \xi))},$$

we have, by the inversion formula of wave packet transform for u_0 and (5) in Lemma 3.2,

$$\begin{aligned} |I_1| &= \frac{1}{\|\varphi_0\|_{L^2}^2} \left| \iiint_{\mathbb{R}^3} \overline{\varphi_0(y - f(0; t, x, \xi))} \varphi_0(y - z) \right. \\ &\quad \left. \times W_{\varphi_0} u_0(z, \eta) e^{iy \cdot (\eta - g(0; t, x, \xi))} dz d\eta dy \right| \\ &\leq \frac{1}{\|\varphi_0\|_{L^2}^2} \iiint_{\mathbb{R}^{3n}} |(1 - \Delta_y)^N \{ \overline{\varphi_0(y - f(0; t, x, \xi))} \varphi_0(y - z) \}| \\ &\quad \times \frac{|W_{\varphi_0} u_0(z, \eta)|}{\langle \eta - g(0; t, x, \xi) \rangle^{2N}} dz d\eta dy \\ &\leq C(1 + |t|)^{2N} \sum_{|\beta_1| + |\beta_2| \leq 2N} \iiint_{\mathbb{R}^{3n}} |\overline{\partial_y^{\beta_1} \varphi_0(y - f(0; t, x, \xi))}| \\ &\quad \times |\partial_y^{\beta_2} \varphi_0(y - z)| \left| \frac{W_{\varphi_0} u_0(z, \eta)}{\langle \eta - \xi \rangle^{2N}} \right| dz d\eta dy. \end{aligned} \tag{19}$$

By Fubini's theorem, we have

$$\begin{aligned} &\| \| I_1 \|_{L_x^\infty} \|_{L_\xi^1} \\ &\leq C(1 + |t|)^{2N} \sum_{|\beta_1| + |\beta_2| \leq 2N} \| \partial_y^{\beta_2} \varphi_0 \|_{L^1} \\ &\quad \times \left\| \left\| \iint_{\mathbb{R}^{2n}} |\partial_y^{\beta_1} \overline{\varphi_0(y - f(0; t, x, \xi))}| \frac{\|W_{\varphi_0} u_0(z, \eta)\|_{L_z^\infty}}{\langle \eta - \xi \rangle^{2N}} d\eta dy \right\|_{L_x^\infty} \right\|_{L_\xi^1} \\ &\leq C(1 + |t|)^{2N} \sum_{|\beta_1| + |\beta_2| \leq 2N} \| \partial_y^{\beta_2} \varphi_0 \|_{L^1} \| \partial_y^{\beta_1} \varphi_0 \|_{L^1} \| \| W_{\varphi_0} u_0(z, \eta) \|_{L_z^\infty} \|_{L_\eta^1} \\ &\leq C_T \| u_0 \|_{M_{\varphi_0}^{\infty, 1}} \end{aligned}$$

for $t \in [0, T]$. Similarly, we have

$$\begin{aligned}
|I_2| &\leq \frac{1}{\|\varphi(\tau, \cdot)\|_{L^2}^2} \sum_{j,k=1}^n \iiint_{\mathbb{R}^{3n}} \left| (1 - \Delta_y)^N \{ \varphi_{jk}(\tau, y - f(\tau; t, x, \xi)) \right. \\
&\quad \times V_{jk}(\tau, f(\tau; t, x, \xi), y) \varphi(\tau, y - z) \} \left| \frac{|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)|}{\langle \eta - g(\tau; t, x, \xi) \rangle^{2N}} dz d\eta dy \right. \\
&\leq \frac{C(1 + |t - \tau|)^{2N}}{\|\varphi(\tau, \cdot)\|_{L^2}^2} \\
&\quad \times \sum_{j,k=1}^n \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} \iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1} \varphi_{jk}(\tau, y - f(\tau; t, x, \xi))| \\
&\quad \times |\partial_y^{\beta_2} V_{jk}(\tau, f(\tau; t, x, \xi), y) \partial_y^{\beta_3} \varphi(\tau, y - z)| \frac{|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)|}{\langle \eta - \xi \rangle^{2N}} dz d\eta dy,
\end{aligned}$$

where $\varphi_{jk}(t, y) = y_j y_k \overline{\varphi(t, y)}$ and V_{jk} is defined by (12). Since

$$|\partial_y^{\beta_2} V_{jk}(\tau, f(\tau; t, x, \xi), y)| \leq C_{\beta_2}$$

for $C_{\beta_2} > 0$, we have

$$\begin{aligned}
\|I_2\|_{L_x^\infty} &\leq \frac{C(1 + |t - \tau|)^{2N}}{\|\varphi(\tau, \cdot)\|_{L^2}^2} \\
&\quad \times \sum_{j,k=1}^n \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} C_{\beta_2} \left\| \iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1} \varphi_{jk}(\tau, y - f(\tau; t, x, \xi))| \right. \\
&\quad \times |\partial_z^{\beta_3} \varphi(\tau, y - z)| \frac{|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)|}{\langle \eta - \xi \rangle^{2N}} dz d\eta dy \left. \right\|_{L_x^\infty} \\
&\leq \frac{C'(1 + T)^{2N}}{\|\varphi(\tau, \cdot)\|_{L^2}^2} \sum_{j,k=1}^n \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} C_{\beta_2} \|\partial_z^{\beta_3} \varphi(\tau, z)\|_{L_z^1} \\
&\quad \times \|\partial_y^{\beta_1} \varphi_{j,k}(\tau, y)\|_{L_y^1} \|\langle \cdot \rangle^{-2N}\|_{L^1} \|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)\|_{L_z^\infty} \|L_\eta^1\| \\
&\leq C'_T \|u(\tau, \cdot)\|_{M_{\varphi(\tau, \cdot)}^{\infty, 1}}
\end{aligned}$$

for $t \in [0, T]$ and $\tau \in [0, t]$. Hence, we have

$$\begin{aligned}
\|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{\infty, 1}} &\leq \|I_1\|_{L_x^\infty} + \int_0^t \|I_2\|_{L_x^\infty} d\tau \\
(20) \quad &\leq C_T \|u_0\|_{M_{\varphi_0}^{\infty, 1}} + C'_T \int_0^t \|u(\tau, \cdot)\|_{M_{\varphi(\tau, \cdot)}^{\infty, 1}} d\tau
\end{aligned}$$

for $t \in [0, T]$. Applying Gronwall's inequality to (20), we obtain

$$(21) \quad \|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{\infty, 1}} \leq C''_T \|u_0\|_{M_{\varphi_0}^{\infty, 1}}$$

for $t \in [0, T]$.

Next, we consider $(p, q) = (1, \infty)$. Take $N \in \mathbb{N}$ satisfying $2N > n$. For all multi-indices β_1 , we have

$$\begin{aligned}
& \|\partial_y^{\beta_1} \varphi(\tau, y - f(\tau; t, x, \xi))\|_{L_x^1} \\
& \leq C(1 + |t - \tau|^2)^{2N} \int_{\mathbb{R}^n} \frac{\langle y - f(\tau; t, x, \xi) \rangle^{2N}}{\langle y - x + (t - \tau)\xi \rangle^{2N}} |\partial_y^{\beta_1} \varphi(\tau, y - f(\tau; t, x, \xi))| dx \\
& \leq C(1 + T^2)^{2N} \left(\sup_{\tau \in [0, T], y \in \mathbb{R}^n} \langle y \rangle^{2N} |\partial_y^{\beta_1} \varphi(\tau, y)| \right) \int_{\mathbb{R}^n} \frac{1}{\langle y - x + (t - \tau)\xi \rangle^{2N}} dx \\
(22) \quad & \leq C_T
\end{aligned}$$

for $t \in [0, T]$ and $\tau \in [0, t]$. Here, we have used (4) in Lemma 3.2. Thus, we have, by (19), (22) and Fubini's theorem,

$$\begin{aligned}
& \| \|I_1\|_{L_x^1} \|_{L_\xi^\infty} \\
& \leq C(1 + |t|)^{2N} \sum_{|\beta_1| + |\beta_2| \leq 2N} \left\| \iiint_{\mathbb{R}^{3n}} \|\partial_y^{\beta_1} \varphi_0(y - f(0; t, x, \xi))\|_{L_x^1} \right. \\
& \quad \times |\partial_y^{\beta_2} \varphi_0(y - z)| \frac{|W_{\varphi_0} u_0(z, \eta)|}{\langle \eta - \xi \rangle^{2N}} dz d\tau \eta dy \Big\|_{L_\xi^\infty} \\
& \leq C'_T \|u_0\|_{M_{\varphi_0}^{1, \infty}}
\end{aligned}$$

for $t \in [0, T]$. In the similar way as above, it follows that

$$\begin{aligned}
& \| \|I_2\|_{L_x^1} \|_{L_\xi^\infty} \\
& \leq \frac{C(1 + |t - \tau|)^{2N}}{\|\varphi(\tau, \cdot)\|_{L^2}^2} \\
& \quad \times \sum_{j, k=1}^n \sum_{|\beta_1| + |\beta_2| + |\beta_3| \leq 2N} \left\| \iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1} \varphi_{j, k}(\tau, y - f(\tau; t, x, \xi))| \right. \\
& \quad \times |\partial_y^{\beta_2} V_{j, k}(\tau, f(\tau; t, x, \xi), y)| \partial_y^{\beta_3} \varphi(\tau, y - z) \frac{|W_{\varphi(\tau, \cdot)} u(\tau, z, \eta)|}{\langle \eta - \xi \rangle^{2N}} dz d\tau \eta dy \Big\|_{L_x^1} \Big\|_{L_\xi^\infty} \\
& \leq C''_T \|u(\tau, \cdot)\|_{M_{\varphi(\tau, \cdot)}^{1, \infty}}
\end{aligned}$$

for $\tau \in [0, t]$ and $t \in [0, T]$. Thus, we have

$$\begin{aligned}
& \|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{1, \infty}} \leq \| \|I_1\|_{L_x^1} \|_{L_\xi^\infty} + \int_0^t \| \|I_2\|_{L_x^1} \|_{L_\xi^\infty} d\tau \\
(23) \quad & \leq C'_T \|u_0\|_{M_{\varphi_0}^{1, \infty}} + C''_T \int_0^t \|u(\sigma, \cdot)\|_{M_{\varphi(\sigma, \cdot)}^{1, \infty}} d\sigma
\end{aligned}$$

for $t \in [0, T]$. Applying Gronwall's inequality to (23), we obtain

$$(24) \quad \|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{1, \infty}} \leq C'''_T \|u_0\|_{M_{\varphi_0}^{1, \infty}}$$

for $t \in [0, T]$.

Finally, we consider the general case. From Theorem 1.1, we have

$$(25) \quad \|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{p,p}} \leq C_T \|u_0\|_{M_{\varphi_0}^{p,p}}.$$

Combing (21), (24) and (25), we have, by the complex interpolation theorem for modulation space,

$$\|u(t, \cdot)\|_{M_{\varphi(t, \cdot)}^{p,q}} \leq \tilde{C}_T \|u_0\|_{M_{\varphi_0}^{p,q}}$$

for $t \in [0, T]$. Therefore we obtain the desired result. \square

APPENDIX A.

First, We remark that if $V(t, x) \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfies (2) for all multi-indices with $|\alpha| \geq 2$ then the ordinary differential equation (3) with the initial condition $f(t) = x$ and $g(t) = \xi$ has a unique solution on \mathbb{R} . In fact, the existence of the solution on $[t - \sigma, t + \sigma]$ for some $\sigma > 0$ is proved by Picard's iteration scheme. Here is the outline. Let $f^{(0)}(s) \equiv x$ and $g^{(0)}(s) \equiv \xi$ and set

$$f^{(k+1)}(s) = x + \int_t^s g^{(k)}(\tau) d\tau \text{ and } g^{(k+1)}(s) = \xi - \int_t^s (\nabla_x V)(\tau, f^{(k)}(\tau)) d\tau.$$

Then $\{f^{(k)}\}$ and $\{g^{(k)}\}$ converge uniformly to some functions $f(s)$ and $g(s)$ on $[t - \sigma, t + \sigma]$ and the functions $f(s)$ and $g(s)$ satisfy the initial value problem and belong to $C^\infty([t - \sigma, t + \sigma])$. Moreover, by using following Lemma A.1, we can show that above fact holds not only on $[t - \sigma, t + \sigma]$ but also on \mathbb{R} , easily. Lemma A.1 is also used in the proof of Lemmas A.2 and A.3.

Lemma A.1. Let $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfy (2) for all multi-indices with $|\alpha| \geq 2$. Then, for all multi-indices β with $|\beta| \geq 1$, $\partial_x^\beta V(t, x)$ is Lipschitz continuous with respect to x , more precisely, there exists $C_\beta > 0$ such that

$$|(\partial_x^\beta V)(t, y) - (\partial_x^\beta V)(t, z)| \leq C_\beta n \|y - z\|_\infty$$

for all $t \in \mathbb{R}$, $y, z \in \mathbb{R}^n$.

Proof. Let $t \in \mathbb{R}$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $z = (z_1, \dots, z_n) \in \mathbb{R}^n$. Since $|y_k - z_k| \leq \|y - z\|_\infty$ for $k = 1, \dots, n$, it is enough to show that

$$|(\partial_x^\beta V)(t, y) - (\partial_x^\beta V)(t, z)| \leq C_\beta \sum_{k=1}^n |y_k - z_k|.$$

Set $F(\theta) = (\partial_x^\beta V)(t, z + \theta(y - z))$. We note that $F(\theta) \in C^\infty([0, 1])$, $F(0) = (\partial_x^\beta V)(t, z)$ and $F(1) = (\partial_x^\beta V)(t, y)$. By the fundamental theorem

of calculus, we have

$$\begin{aligned}
 (\partial_x^\beta V)(t, y) - (\partial_x^\beta V)(t, z) &= \int_0^1 \frac{d}{d\theta} F(\theta) d\theta \\
 (26) \qquad \qquad \qquad &= \int_0^1 \sum_{k=1}^n (y_k - z_k) (\partial_{x_k} \partial_x^\beta V)(t, z + \theta(y - z)) d\theta.
 \end{aligned}$$

Since V satisfies (2) for $|\alpha| \geq 2$, we obtain

$$(27) \qquad \qquad \qquad |(\partial_{x_k} \partial_x^\beta V)(t, z + \theta(y - z))| \leq C_\beta$$

for $k = 1, 2, \dots, n$. Combining (26) and (27), we obtain the desired result. \square

Next, we establish one more lemma relating to the Lemma A.3.

Lemma A.2. Let $h \in \mathbb{R} \setminus \{0\}$, $T > 0$ and $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfy (2) for all multi-indices α with $|\alpha| \geq 2$. Suppose that $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are solutions to (3) with $f(t) = x$ and $g(t) = \xi$. For $k = 1, \dots, n$, we set

$$\begin{aligned}
 \phi_{h,k}(s; t, x, \xi) &= (f(s; t, x + he_k, \xi) - f(s; t, x, \xi), g(s; t, x + he_k, \xi) - g(s; t, x, \xi))
 \end{aligned}$$

and

$$\begin{aligned}
 \psi_{h,k}(s; t, x, \xi) &= (f(s; t, x, \xi + he_k) - f(s; t, x, \xi), g(s; t, x, \xi + he_k) - g(s; t, x, \xi)),
 \end{aligned}$$

where $e_k = (0, \dots, 0, \overset{k}{1}, 0, \dots, 0) \in \mathbb{R}^n$. Then

$$\lim_{h \rightarrow 0} \sup_{|s-t| \leq T} \|\phi_{h,k}(s; t, x, \xi)\|_\infty = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \sup_{|s-t| \leq T} \|\psi_{h,k}(s; t, x, \xi)\|_\infty = 0.$$

Proof. Here, we only show that $\lim_{h \rightarrow 0} \sup_{|s-t| \leq T} \|\phi_{h,1}(s; t, x, \xi)\|_\infty = 0$. We can treat the other cases in the same way. Put

$$F(s; t, x, \xi) = f(s; t, x + he_1, \xi) - f(s; t, x, \xi)$$

and

$$G(s; t, x, \xi) = g(s; t, x + he_1, \xi) - g(s; t, x, \xi).$$

Since $F(t; t, x, \xi) = f(t; t, x + he_1, \xi) - f(t; t, x, \xi) = he_1$ and

$$\frac{d}{ds} F(s; t, x, \xi) = g(s; t, x + he_1, \xi) - g(s; t, x, \xi) = G(s; t, x, \xi),$$

we have

$$F(s; t, x, \xi) = he_1 + \int_t^s G(\tau; t, x, \xi) d\tau.$$

Thus, we have

$$(28) \qquad \|F(s; t, x, \xi)\|_\infty \leq |h| + \left| \int_t^s \|G(\tau; t, x, \xi)\|_\infty d\tau \right|.$$

On the other hand, since $G(t; t, x, \xi) = 0$ and

$$\frac{d}{ds}G(s; t, x, \xi) = -\nabla_x V(s, f(s; t, x + he_1, \xi)) + \nabla_x V(s, f(s; t, x, \xi)),$$

we have

$$G(s; t, x, \xi) = - \int_t^s \left\{ \nabla_x V(\tau, f(\tau; t, x + he_1, \xi)) - \nabla_x V(\tau, f(\tau; t, x, \xi)) \right\} d\tau.$$

By Lemma A.1, there exists $C > 0$ such that

$$\begin{aligned} & \|G(s; t, x, \xi)\|_\infty \\ & \leq \left| \int_t^s \|\nabla_x V(\tau, f(\tau; t, x + he_1, \xi)) - \nabla_x V(\tau, f(\tau; t, x, \xi))\|_\infty d\tau \right| \\ (29) \quad & \leq Cn \left| \int_t^s \|F(\tau; t, x, \xi)\|_\infty d\tau \right|. \end{aligned}$$

From (28) and (29), we have

$$(30) \quad \|\phi_{h,1}(s; t, x, \xi)\|_\infty \leq |h| + C' \left| \int_t^s \|\phi_{h,1}(\tau, t, x, \xi)\|_\infty d\tau \right|,$$

where $C' = \max\{1, Cn\}$. Since $|s-t| \leq T$, applying Gronwall's inequality to (30) gives

$$(31) \quad \|\phi_{h,1}(s; t, x, \xi)\|_\infty \leq |h| e^{C'|s-t|} \leq |h| e^{C'T}.$$

Hence, we obtain the desired result. \square

Next, we show the differentiability of the solution to (3) in initial datum.

Lemma A.3. Let $T > 0$, $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfy (2) for all multi-indices α with $|\alpha| \geq 2$. Suppose that $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are solutions to (3) satisfying $f(t) = x$ and $g(t) = \xi$. Then $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are C^∞ -function with respect to x and ξ in $|s-t| \leq T$.

Proof. Let $A(s, y)$ be the $2n \times 2n$ matrix defined by

$$(32) \quad A(s, y) = \begin{pmatrix} O_n & -\frac{\partial^2}{\partial x_1 \partial x_1} V(s, y) & \cdots & -\frac{\partial^2}{\partial x_1 \partial x_n} V(s, y) \\ & \vdots & \ddots & \vdots \\ & -\frac{\partial^2}{\partial x_n \partial x_1} V(s, y) & \cdots & -\frac{\partial^2}{\partial x_n \partial x_n} V(s, y) \\ E_n & & & O_n \end{pmatrix},$$

where O_n is the $n \times n$ zero matrix and E_n is the $n \times n$ identity matrix. Let $h \in \mathbb{R} \setminus \{0\}$ and put

$$\begin{aligned} \phi_{h,j}(s; t, x, \xi) \\ = (f(s; t, x + he_j, \xi) - f(s; t, x, \xi), g(s; t, x + he_j, \xi) - g(s; t, x, \xi)) \end{aligned}$$

and

$$\begin{aligned} \psi_{h,j}(s; t, x, \xi) \\ = (f(s; t, x, \xi + he_j) - f(s; t, x, \xi), g(s; t, x, \xi + he_j) - g(s; t, x, \xi)), \end{aligned}$$

where $e_j = (0, \dots, 0, \overset{j}{1}, 0, \dots, 0) \in \mathbb{R}^n$ and $j = 1, 2, \dots, n$. Suppose that

$$w^{(k)}(s; t, x, \xi) = (w_{1,k}(s; t, x, \xi), \dots, w_{2n,k}(s; t, x, \xi))$$

is the solution of

$$(33) \quad \begin{cases} \frac{dw(s)}{ds} = w(s)A(s, f(s; t, x, \xi)), \\ w(t) = (0, \dots, 0, \overset{k}{1}, 0, \dots, 0), \end{cases}$$

where $k = 1, 2, \dots, 2n$.

First, we show that

$$(34) \quad \lim_{h \rightarrow 0} \sup_{|s-t| \leq T} \left\| \frac{\phi_{h,j}(s; t, x, \xi)}{h} - w^{(j)}(s; t, x, \xi) \right\|_{\infty} = 0.$$

From (3) and (26), it is easy to see that

$$\begin{aligned} & \frac{d}{ds} \left(\frac{\phi_{h,j}(s; t, x, \xi)}{h} \right) \\ &= \frac{1}{h} \int_0^1 \phi_{h,j}(s, t, x, \xi) \\ & \quad \times A(s, f(s; t, x + he_j, \xi) + \theta(f(s; t, x, \xi) - f(s; t, x + he_j, \xi))) d\theta \\ (35) \quad &= \frac{1}{h} \phi_{h,j}(s, t, x, \xi) A(s, f(s; t, x, \xi)) + \gamma_{h,j}(s, t, x, \xi), \end{aligned}$$

where

$$\begin{aligned} (36) \quad \gamma_{h,j}(s; t, x, \xi) &= \frac{1}{h} \int_0^1 \phi_{h,j}(s, t, x, \xi) \left\{ A(s, f(s; t, x + he_j, \xi) \right. \\ & \quad \left. + \theta(f(s; t, x, \xi) - f(s; t, x + he_j, \xi))) - A(s, f(s; t, x, \xi)) \right\} d\theta. \end{aligned}$$

By the definition of $A(s, y)$ and Lemma A.1, there exists $C > 0$ such that

$$\begin{aligned}
 & \left\| A\left(s, f(s; t, x + he_j, \xi) + \theta(f(s; t, x, \xi) - f(s; t, x + he_j, \xi))\right) \right. \\
 & \quad \left. - A(s, f(s; t, x, \xi)) \right\|_{\infty} \\
 & \leq Cn(1 - \theta) \|f(s; t, x + he_j, \xi) - f(s; t, x, \xi)\|_{\infty} \\
 (37) \quad & \leq Cn \|\phi_{h,j}(s; t, x, \xi)\|_{\infty}
 \end{aligned}$$

for $\theta \in [0, 1]$. Thus (31), (36) and (37) yield

$$\begin{aligned}
 \|\gamma_{h,j}(s; t, x, \xi)\|_{\infty} & \leq \frac{2Cn^2}{|h|} \|\phi_{h,j}(s; t, x, \xi)\|_{\infty}^2 \\
 (38) \quad & \leq 2Cn^2 \|\phi_{h,j}(s; t, x, \xi)\|_{\infty} e^{C'|s-t|}
 \end{aligned}$$

for $C' > 0$. As V satisfies the estimate (2) for all multi-indices α with $|\alpha| \geq 2$, there exists $M > 0$ such that

$$(39) \quad \|A(s, f(s; t, x, \xi))\|_{\infty} \leq M.$$

Since $\phi_{h,j}(t; t, x, \xi)/h - w^{(j)}(t; t, x, \xi) = 0$, we have, by (33) and (35),

$$\begin{aligned}
 & \frac{\phi_{h,j}(s; t, x, \xi)}{h} - w^{(j)}(s; t, x, \xi) \\
 & = \int_t^s \left(\frac{1}{h} \frac{d}{d\tau} \phi_{h,j}(\tau; t, x, \xi) - \frac{d}{d\tau} w^{(j)}(\tau; t, x, \xi) \right) d\tau \\
 & = \int_t^s \gamma_{h,j}(\tau; t, x, \xi) d\tau \\
 & \quad + \int_t^s \left(\frac{\phi_{h,j}(\tau; t, x, \xi)}{h} - w^{(j)}(\tau; t, x, \xi) \right) A(\tau, f(\tau; t, x, \xi)) d\tau.
 \end{aligned}$$

So we have, by (38) and (39),

$$\begin{aligned}
 & \left\| \frac{\phi_{h,j}(s; t, x, \xi)}{h} - w^{(j)}(s; t, x, \xi) \right\|_{\infty} \\
 & \leq 2Cn^2 e^{C'T} \int_{t-T}^{t+T} \|\phi_{h,j}(\tau; t, x, \xi)\|_{\infty} d\tau \\
 (40) \quad & \quad + 2nM \left| \int_t^s \left\| \frac{\phi_{h,j}(\tau; t, x, \xi)}{h} - w^{(j)}(\tau; t, x, \xi) \right\|_{\infty} d\tau \right|
 \end{aligned}$$

for $|s - t| \leq T$. Applying Gronwall's inequality to (40), we obtain

$$\begin{aligned}
 & \left\| \frac{\phi_{h,j}(s; t, x, \xi)}{h} - w^{(j)}(s; t, x, \xi) \right\|_{\infty} \\
 & \leq 2Cn^2 e^{(2nM+C')T} \int_{t-T}^{t+T} \|\phi_{h,j}(\tau; t, x, \xi)\|_{\infty} d\tau
 \end{aligned}$$

for $|s - t| \leq T$. By Lemma A.2, we obtain (34). Thus, we have

$$\frac{\partial f_l(s; t, x, \xi)}{\partial x_k} = w_{l,k}(s; t, x, \xi) \in C(\mathbb{R})$$

and

$$\frac{\partial g_l(s; t, x, \xi)}{\partial x_k} = w_{n+l,k}(s; t, x, \xi) \in C(\mathbb{R})$$

for $k, l = 1, \dots, n$.

On the other hand, in the similar calculation as above, we have

$$\lim_{h \rightarrow 0} \sup_{|s-t| \leq T} \left\| \frac{\psi_{h,j}(s; t, x, \xi)}{h} - w^{(n+j)}(s; t, x, \xi) \right\|_{\infty} = 0.$$

Thus,

$$\frac{\partial f_l(s; t, x, \xi)}{\partial \xi_k} = w_{l,n+k}(s; t, x, \xi) \in C(\mathbb{R})$$

and

$$\frac{\partial g_l(s; t, x, \xi)}{\partial \xi_k} = w_{n+l,n+k}(s; t, x, \xi) \in C(\mathbb{R})$$

for $k, l = 1, \dots, n$. Hence, $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are C^1 -function with respect to x and ξ .

Using above fact, we can easily show, by induction, that if $V(t, x)$ is C^{r+1} -function in x then $f(s; t, x, \xi)$ and $g(s; t, x, \xi)$ are C^r -function with respect to x and ξ . Therefore we obtain the desired result. \square

Proof of Lemma 3.1. Put $w^{(k)}(s; t, x, \xi) = (w_{1,k}, w_{2,k}, \dots, w_{2n,k})$ for $1 \leq k \leq 2n$. By Lemma A.3, $w^{(k)}(s)$ are the solutions of

$$\begin{cases} \frac{dw(s)}{ds} = w(s)A(s, f(s; t, x, \xi)), \\ w(t) = (0, \dots, 0, \underbrace{1}_k, 0, \dots, 0), \end{cases}$$

where $A(s, f(s; t, x, \xi)) = (a_{ij})$ is defined by (32). Then

$$(41) \quad \frac{d(\det M(s; t, x, \xi))}{ds} = \begin{vmatrix} \frac{dw_{1,1}}{ds} & \cdots & \frac{dw_{1,2n}}{ds} \\ w_{2,1} & \cdots & w_{2,2n} \\ \vdots & & \vdots \\ w_{2n,1} & \cdots & w_{2n,2n} \end{vmatrix} + \cdots + \begin{vmatrix} w_{1,1} & \cdots & w_{1,2n} \\ w_{2,1} & \cdots & w_{2,2n} \\ \vdots & & \vdots \\ \frac{dw_{2n,1}}{ds} & \cdots & \frac{dw_{2n,2n}}{ds} \end{vmatrix}.$$

Since $\frac{dw^{(k)}(s)}{ds} = w^{(k)}(s)A(s, f(s; t, x, \xi))$, we have $\frac{dw_{1,k}(s)}{ds} = \sum_{j=1}^{2n} a_{jl}w_{j,k}$ and then

$$(42) \quad \begin{vmatrix} \frac{dw_{1,1}}{ds} & \cdots & \frac{dw_{1,2n}}{ds} \\ w_{2,1} & \cdots & w_{2,2n} \\ \vdots & & \vdots \\ w_{2n,1} & \cdots & w_{2n,2n} \end{vmatrix} = \sum_{j=1}^{2n} a_{j1} \begin{vmatrix} w_{j,1} & \cdots & w_{j,2n} \\ w_{2,1} & \cdots & w_{2,2n} \\ \vdots & & \vdots \\ w_{2n,1} & \cdots & w_{2n,2n} \end{vmatrix} = a_{11} \det M(s).$$

From (41) and (42), we have

$$\frac{d(\det M(s; t, x, \xi))}{ds} = (a_{11} + \cdots + a_{2n,2n}) \det M(s; t, x, \xi).$$

Since $\text{tr} A(s, f(s; t, x, \xi)) = 0$, we have $\frac{d(\det M(s; t, x, \xi))}{ds} = 0$. Therefore

$$\det M(s; t, x, \xi) = \det M(t; t, x, \xi) = \det E_{2n} = 1. \quad \square$$

REFERENCES

- [1] Á. Bényi, K. Gröchenig, K. Okoudjou and L.G. Rogers, *Unimodular Fourier multipliers for modulation spaces*, J. Funct. Anal. 246 (2007), pp. 366–384.
- [2] Á. Bényi, K. Okoudjou, *Local well-posedness of nonlinear dispersive equations on modulation spaces*, Bull. Lond. Math. Soc. 41 (2009), pp. 549–558.
- [3] E. Cordero and F. Nicola, *Metaplectic representation on Wiener amalgam spaces and applications to the Schrödinger equation*, J. Funct. Anal. 254 (2008), pp. 506–534.
- [4] E. Cordero and F. Nicola, *Strichartz estimates in Wiener amalgam spaces for the Schrödinger equation*, Math. Nachr. 281 (2008), pp. 25–41.
- [5] A. Córdoba and C. Fefferman, *Wave packets and Fourier integral operators*, Comm. Partial Differential Equations 3 (1978), pp. 979–1005.
- [6] H. G. Feichtinger, *Modulation spaces on locally compact abelian groups*, in: M. Krishna, R. Radha and S. Thangavelu (Eds.), *Wavelets and their Applications*, Chennai, India, Allied Publishers, New Delhi, 2003, pp. 99–140, Updated version of a technical report, University of Vienna, 1983.
- [7] D. Fujiwara, *A construction of the fundamental solution for the Schrödinger equation*, J. Analyse Math. 35 (1979), pp. 41–96.
- [8] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, 2001.
- [9] K. Kato, M. Kobayashi and S. Ito, *Representation of Schrödinger operator of a free particle via short time Fourier transform and its applications*, Tohoku Math. J. 64 (2012), pp. 223–231.
- [10] K. Kato, M. Kobayashi and S. Ito, *Remark on wave front sets of solutions to Schrödinger equation of a free particle and a harmonic oscillator*, SUT J. Math. 47 (2011), pp. 175–183.
- [11] K. Kato, M. Kobayashi and S. Ito, *Remarks on Wiener Amalgam space type estimates for Schrödinger equation*, pp. 41–48, RIMS Kôkyûroku Bessatsu, B33, Res. Inst. Math. Sci. (RIMS), Kyoto, 2012.
- [12] M. Kobayashi and M. Sugimoto, *The inclusion relation between Sobolev and modulation spaces*, J. Funct. Anal. 260 (2011), pp. 3189–3208.
- [13] A. Miyachi, F. Nicola, S. Rivetti, A. Tabacco and N. Tomita, *Estimates for unimodular Fourier multipliers on modulation spaces*, Proc. Amer. Math. Soc. 137 (2009), pp. 3869–3883.

- [14] N. Tomita, *Unimodular Fourier multipliers on modulation spaces $M^{p,q}$ for $0 < p < 1$.*, Harmonic analysis and nonlinear partial differential equations, pp. 125–131, RIMS Kôkyûroku Bessatsu, B18, Res. Inst. Math. Sci. (RIMS), Kyoto, 2010.
- [15] B. Wang and C. Huang, *Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations*, J. Differential Equations 239 (2007), pp. 213–250.
- [16] B. Wang and H. Hudzik, *The global Cauchy problem for the NLS and NLKG with small rough data*, J. Differential Equations 232 (2007), pp. 36–73.
- [17] B. Wang, L. Zhao and B. Guo, *Isometric decomposition operators, function spaces $E_{p,q}^\lambda$ and applications to nonlinear evolution equations*, J. Funct. Anal. 233 (2006), pp. 1–39.

DEPARTMENT OF MATHEMATICS
TOKYO UNIVERSITY OF SCIENCE
KAGURAZAKA 1-3, SHINJUKU-KU, TOKYO 162-8601, JAPAN
E-mail address: kato@ma.kagu.tus.ac.jp

DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE
YAMAGATA UNIVERSITY
KOJIRAKAWA 1-4-12, YAMAGATA-CITY, YAMAGATA 990-8560, JAPAN
E-mail address: kobayashi@sci.kj.yamagata-u.ac.jp

DEPARTMENT OF MATHEMATICS
TOKYO UNIVERSITY OF SCIENCE
KAGURAZAKA 1-3, SHINJUKU-KU, TOKYO 162-8601, JAPAN
E-mail address: ito@ma.kagu.tus.ac.jp